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NONLINEAR COST NETWORK MODELS IN TRANSPORTATION ANALYSIS

by

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ABSTRACT

The analysis of transportation phenomena by quantitative approaches naturally gives rise to network models that represent the spatial characteristics of the transport infrastructure. This paper surveys the nonlinear cost models that arise in transportation analysis and points out the principal methods used for their solution in practice. The problems presented are the network equilibrium problem, the problem of finding an origin/destination matrix, the spatial price equilibrium problem and certain concave cost network flow problems.

RÉSUMÉ

L'analyse des phénomènes de transport par des approches quantitatives résulte naturellement dans des modèles de réseau qui décrivent les caractéristiques spatiales de l'infrastructure des transports. Cet article résume les modèles à coût non-linéaire utilisés dans l'analyse des transports et indique les principales méthodes utilisées pour résoudre ces problèmes en pratique. Les problèmes présentés dans le texte sont le problème d'équilibre de réseau, le problème d'estimation d'une matrice origine/destination et certains problèmes de flots à coûts concaves.

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1. INTRODUCTION

The analysis of transportation phenomena by quantitative approaches naturally gives rise to network models that represent the spatial characteristics of the transport infrastructure. The issues of interest that generate the need to formulate, analyze, solve and apply network models in this context may be addressed by determining the way in which people, vehicles and goods make or should make use of an existing or proposed transport infrastructure. To this end, it is pertinent to formulate models of the demand for transportation and models that determine the way in which this demand makes use of existing or proposed infrastructure. Such models may be descriptive, in attempting to reproduce observed behavior patterns or may be normative, in attempting to prescribe ways in which the transportation infrastructure should be used in an efficient way.

The practical contexts which give rise to such models are varied and the stimulus for these models arise from urban and regional contexts, as well as national and international situations. Whenever congestion phenomena are present, the cost functions that are employed to reflect such situations are nonlinear; in most applications they are convex or monotone. Whenever economies of scale phenomena are present the cost functions are naturally concave. In most cases, the resulting models may be formulated as nonlinear cost optimization models over an appropriate network. Other versions of these problems may be formulated as nonlinear complementarity problems or variational inequalities over a network.

The purpose of this paper is to survey these nonlinear cost network models that arise in transportation analysis and point out the principal methods used for their solution in practice. The current state of knowledge in this field has seen great progress in the past twenty years, stimulated in part by the interest that researchers with mathematical programming interest have taken in these problems and by the advent of digital computers, which has made the solution of the models accessible to practitioners. In spite of the recent progress, the study of nonlinear cost models in transportation analysis still poses significant challenges in the formulation of models, analysis and development of solution algorithms.

The paper is organized in the following way. First the network equilibrium problem is presented and the various contexts in which it is applied are described. Then the problem of estimating an origin/destination trip matrix is described as well as all the variants which are generally refered to as spatial interaction models. The spatial price equilibrium problem, which occurs in interregional trade solutions, is considered next. Finally, we consider problems with concave cost structure which arise in location problems and transportation problems where quantity discounts are available.

No attempt has been made to cover this large area of research and application exhaustively; rather, the main classes of models and solution approaches have been covered.

2. THE NETWORK EQUILIBRIUM PROBLEM

Traffic equilibria models are descriptive models that are aimed at predicting link flows and travel times that result from the way in which users choose routes from their origins to their destinations in a transportation network. The behavioral assumption used as been stated by Wardrop (1951), postulates that "the journey time on all the routes actually used are equal and less than those which would be experienced by a single vehicle on any unused route". The implication of this principle is that the results actually used are the shortest under prevailing traffic conditions and their perception by the user. This characterization of equilibrium of route choice, or traffic equilibrium, which is identical to the ideas enunciated by Knight (1924), became accepted over the past 30 years as a sound behavioral principle to describe the subdivision of trips over alternative routes. The traffic flows that satisfy this principle are usually referred to as "user-optimized" flows, since each user chooses the route that he perceives to be the best. By contrast "system-optimized" flows are characterized by Wardrop's normative principle which states that "the average journey time is minimum". Beckmann, McGuire and Winsten (1956) were the first to formulate and analyze as model of traffic equilibrium in a network. Their seminal work was the starting point for contributions to follow in this area which were achieved mostly within the past fifteen years. The presentation of the model below is inspired by the work of Smith (1979), Aashtiani and Magnanti (1981) and Dafermos (1980, 1981).

Consider a transportation network which permits the flow of one type of traffic (vehicles or passengers) on its links. The nodes n, $n \in N$, represent origins, destinations and intersections of links; the links a, $a \in A$, represent the transportation infrastructure. The flow of trips on link a is v_a and the cost of travelling on a link is given by a user cost function $s_a(v)$, where v is the vector of link flows over the entire network. This cost function may model the time delay for travel on that arc, in which case it is commonly refered to as a volume/delay function, however it may model other costs, such as fuel consumption. The vector user cost function $s(v)$ is assumed to be monotone

$$(s(v')-s(v''))^T(v'-v'') \geq 0, \qquad \forall\, v', v'' \text{ feasible} \tag{1}$$

continuous and differentiable.

The origin to destination demands, g_i, $i \in I$, where I is the set of origin/destination (O/D) pairs, may use directed paths k, $k \in K_i$, where K_i is the set of paths for O/D pair i ($K_i \neq \phi$). The flows on paths k, h_k, satisfy conservation of flow and nonnegativity

$$\left.\begin{aligned} \sum_{k \in K_i} h_k &= g_i, \qquad i \in I \\ \text{and} \qquad h_k &\geq 0, \qquad k \in K_i,\ i \in I \end{aligned}\right\} \tag{2}$$

The corresponding link flows v_a are given by

$$v_a = \sum_{i \in I} \sum_{k \in K_i} \delta_{ak} \cdot h_k, \qquad a \in A \tag{3}$$

where $\delta_{ak} = \begin{cases} 1 & \text{if link a belongs to path k} \\ 0 & \text{otherwise} \end{cases}$

The cost of each path $s_k(=s_k(v))$ is the sum of the user costs of the links in the path

$$s_k = \sum_a \delta_{ak} \cdot s_a(v), \qquad k \in K_i, \; i \in I \tag{4}$$

Let $u_i(=u_i(v))$ be the cost of the least cost path for any O/D pair i

$$u_i = \min_{k \in K_i} s_k, \qquad i \in I \tag{5}$$

The demands g_i are given by functions $G_i(u)$, where u is the vector of least cost travel times for all the O/D pairs of the network

$$g_i = G_i(u), \qquad i \in I \tag{6}$$

The vector demand function $G(u)$ is assumed to be strictly antitone

$$(G(u') - G(u''))^T (u' - u'') < 0, \qquad u' \neq u'' \tag{7}$$

continuous, differentiable and bounded from above.

The network equilibrium model is formulated by supposing that for every O/D pair, Wardrop's user optimal principle is satisfied, that is

$$s_k^* - u_i^* \begin{cases} = 0 & \text{if } h_k^* > 0 \\ \geq 0 & \text{if } h_k^* = 0 \end{cases} \qquad k \in K_i, \; i \in I \tag{8}$$

over the feasible set (2)-(6).

It is relatively straightforward to show that (8) may be restated in the "complementarity" form

$$u_i^* \le s_k^* \quad \text{and} \quad (s_k^* - u_i^*)h_k^* = 0, \qquad k \in K_i, \quad i \in I \tag{9}$$

and that (8) and (5) are equivalent to

$$s_{k_1}^* \le s_{k_2}^*, \quad \text{if } h_{k_1}^* > 0, \quad k_1, k_2 \in K_i, \quad i \in I \tag{10}$$

Thus, Wardrop's principle may be stated mathematically in several forms.

The "complementarity form" (9) statement has been used by Aashtiani and Magnanti to formulate the network equilibrium problems as a nonlinear complementarity problem by letting

$$F(x) = \{(s_k - u_i),\ k \in K_i,\ i \in I;\ (\sum_{k \in K_i} h_k - G_i(u)),\ i \in I\} \tag{11}$$

and $x = (h,u)$. Then the network equilibrium problem is equivalent to solving $F(x^*) \cdot x^* = 0$, $F(x^*) \ge 0$, $x^* \ge 0$. The condition $(\sum_{k \in K_i} h_k - G_i(u))u_i = 0$, $i \in I$, does not correspond directly with the original statement of the problem, however under the reasonable assumptions that $G_i(u) > 0$, $i \in I$ and $s_k > 0$, $k \in K_i$, $i \in I$ it is equivalent to (2).

This formulation may be used to derive existence results. A solution (h,u) exists if the cost functions $s_a(v)$, $a \in A$ are continuous and positive and if $G_i(u), I \in I$ are continuous nonnegative functions bounded from above. These conditions are relatively weak and are certainly satisfied by most cost and demand functions which arise in practice (see Aashtiani and Magnanti, 1981). Uniqueness results may be derived as well.

Another, more useful restatement of the network equilibrium problem is in the form of a variational inequality. This may be done by direct

derivation or by applying the result (see Kinderlehrer and Stampacchia (1980)) that $x^* \in R^+$ is a solution to the complementarity problem if and only if x^* is a solution of the variational inequality $F(x^*)(x-x^*) \geq 0$, $x \in R^+$. The variational inequality that results by applying this equivalence is

$$\sum_{i \in I} \sum_{k \in K_i} (s_k^* - u_i^*)(h_k - h_k^*) - \sum_{i \in I} G_i(u^*)\,(u_i - u_i^*) \geq 0 \qquad (12)$$

$$h_k \geq 0,\ k \in K_i,\ i \in I$$

$$u_i \geq 0,\ i \in I$$

(see also Fisk and Boyce (1982), Magnanti (1982)).

When the demand is constant $G_i(u) = \bar{g}_i$, $i \in I$, (12) reduces to the variational inequality formulation of the problem derived by Smith (1979)

$$s(v^*)^T(v - v^*) \geq 0, \qquad (13)$$

and since the demand function is invertible, that is $W_i(g) = u_i$, $i \in I$, or in vector notation $W(g) = u$, (12) reduces to the variational inequality derived by Dafermos (1981)

$$s(v^*)^T(v - v^*) - W(g^*)^T(g - g^*) \geq 0 \qquad (14)$$

The usefulness of the variational inequality formulation is that, (13) and (14) may be easier to solve since the network equilibrium problem is formulated in terms of link flows rather than path flows.

An important property of a network model that is used to simulate traffic flows for different future network configurations, is that for a given scenario the flows and origin to destination costs be unique. Unless this were so, the comparison of the different future situations would be

difficult to carry out; differences between scenarios would depend on which solution was found for each case as well as on the different network configurations. Fortunately, for many situations of interest, the network equilibrium model has unique flows, demand and origin to destination costs. This is ensured when the user cost function is strictly monotone and the demand function (and its inverse) is strictly antitone, as assumed in the model formulation.

To demonstrate this (see Asmuth (1978), Smith (1979) and Aashtiani and Magnanti (1981)) suppose that there exist two distinct solutions (v',g') and (v",g"). By writing (14) once with v=v', g=g', and v*=v", g*=g" and once with v=v", g=g" and v*=v', g*=g', one obtains by adding the two inequalities

$$(s(v')-s(v"))^T(v'-v") - (W(g')-W(g"))^T(g'-g") \leq 0 \qquad (15)$$

However by (1) and (7) it follows that (15) is satisfied if and only if each term is equal to zero. Thus

$$(s(v')-s(v"))^T(v'-v") = 0, \qquad (16)$$

and

$$(W(g')-W(g"))^T(g'-g") = 0 \quad . \qquad (17)$$

It follows then from (17) that g'=g", and from (16) that $s_a(v')=s_a(v")$, hence the link user costs are unique as well. Since u_i^*, $i \in I$ are the lengths of shortest paths computed for each O/D pair i based on the user costs $s_a(v^*)$, it follows that they are unique as well. We note that if the user cost function is strictly monotone, then the link flows are unique as well.

A uniqueness result that does not require the invertibility of the demand functions may be obtained by using (12). The demands are unique if the demand functions are strictly antitone, however the path flows are not necessarily unique. The nonuniqueness of the path flows is of importance in applications of the model. Given the equilibrium flows v_a^*, the corresponding path flows are given as the solution of the simultaneous linear equations

$$v_a^* = \sum_{i \in I} \sum_{k \in K_i} \delta_{ak} h_k^* , \qquad a \in A \tag{18}$$

Usually, the number of paths used is much larger than the number of links of the network, and in (18) the number of variables h_k^* far exceeds the number of constraints, which equals to $|A|$. The consequence of this fact is that, in the analysis of particular network configuration, the paths used may be obtained - however they do not represent reliable information, since they are not necessarily unique.

Another important property of the network equilibrium problem is that it is stable when the cost function is monotone. The equilibrium traffic flows depend continuously upon the travel demands and travel cost functions. Dafermos and Nagurney (1982) demonstrated, by using the variational inequality formulation (13) that a small change in the travel costs induces a small change in the equilibrium flows, that is

$$\|v^* - v\| \leq \frac{1}{\alpha} \|s(v^*) - s(v)\| \tag{19}$$

where α is the constant used in the definition of the strong monotonicity condition on the cost functions $s(v)$.

Most of the applications of the network equilibrium model in practice have been achieved for simpler versions of the model (14) subject to (2)-(3). The user cost functions that have been calibrated in practice are functions of only the flows on the link, that is $s_a(v) = s_a(v_a)$, $a \in A$ and the demand functions depend only on the travel cost between the origin and the destination of the node pair i, that is $G_i(u) = G_i(u_i)$. Thus, the requirements on the cost functions are now to a requirement that the user cost functions are all continuous and increasing and that the demand functions are continuous, strictly decreasing and bounded from above. (14) may now be restated as

$$\sum_a s_a(v_a^*)(v_a - v_a^*) - \sum_{i \in I} W_i(g_i^*)(g_i - g_i^*) \geq 0 \qquad (20)$$

With these separability conditions, (20) is equivalent to the convex cost optimization problem

$$\text{Min } Z = \sum_{a \in A} \int_0^{v_a} s_a(x)dx - \sum_{i \in I} \int_0^{g_i} W_i(y)dy \qquad (21)$$

subject to (2) - (3). Since the user cost functions are increasing and the demand functions are strictly decreasing, the objective function (21) is convex and (20) may be interpreted as the first order optimality condition, that the directional derivative of Z is nonnegative. A rigorous proof of this equivalence may be found in Kinderlehrer and Stampacchia (1980). This is the classical result found by Beckmann et al. (1956). The Kuhn-Tucker conditions of the convex cost optimization problem (21) subject to (1) - (3) are equivalent to the statement of Wardrop's user optimal principle of route choice in the form (8). The existence of the

equivalent optimization problem greatly facilitated the use of the model in practice, since a number of efficient algorithms are available for solving such problems.

From the large repertory of methods that are available for solving convex cost minimization problems subject to linear constraints, the adaptation of the linear approximation method of Frank and Wolfe (1956) and some ot its variants are the most commonly used algorithms. Bruynooghe, Gibert and Sakarovitch (1968) were the first to propose the method, however the subsequent work of LeBlanc, Morlok and Pierskalla (1975), Golden (1975), Nguyen (1976), Dow and Van Vliet (1979) made this method popular in practice. In the context of "packet switching" networks, the same algorithm was also proposed by Fratta, Gerla and Kleinrock (1973). Computer codes are widely available for solving the fixed demand version of this problem, that is when $g_i = \bar{g}_i$, $i \in I$. We proceed to present the details of the adaptation of this method to solve the fixed demand network equilibrium problem and then discuss some of its variants, both for fixed and variable demand.

We consider now the fixed demand problem

$$\text{Min } Z = \sum_{a \in A} \int_0^{v_a} s_a(x)dx \tag{22}$$

subject to

$$\sum_{k \in K_i} h_k = \bar{g}_i, \quad i \in I \tag{23}$$

$$h_k \geq 0, \quad k \in K_i, \ i \in I \tag{24}$$

and we recall (3), that is $v_a = \sum_{i \in I} \sum_{k \in K_i} \delta_{ak} \cdot h_k, \quad a \in A.$

Starting from an initial feasible solution, the linear approximation method obtains a feasible direction by linearizing the objective function, solving a linear programming subproblem and then finding an improved solution on the link segment between the current solution and the solution of the subproblem. For the network equilibrium problem the linearized subproblem at iteration ℓ is

$$\text{Min} \sum_{i\in I} \sum_{k\in K_i} \sum_{a\in A} s_a(v_a^{\ell})\delta_{ak} \cdot f_k \tag{25}$$

$$\text{subject to} \quad \sum_{k\in K_i} f_k = \bar{g}_i, \quad i\in I \tag{26}$$

$$f_k \geq 0, \quad k\in K_i, \quad i\in I \tag{27}$$

By changing the order of summation in (25) and by using (4), the objective becomes

$$\text{Min} \sum_{i\in I} \sum_{k\in K_i} s_k^{\ell} \cdot f_k \tag{28}$$

The solution of this problem is obtained trivially by computing shortest paths for each O/D pair i and allocating the demand $\bar{g}_i$, to that path. Note that the complementarity slackness conditions for the dual constraints of this linear program are

$$f_k(u_i - s_k^{\ell}) = 0, \quad k\in K_i, \quad i\in I \tag{29}$$

and dual constraints are

$$u_i \leq s_k^{\ell}, \quad k\in K_i, \quad i\in I \ . \tag{30}$$

The optimal u_i are the length of the shortest paths based on the current link user costs $s_a(v_a^\ell)$ and (29) is a statement of Wardrop's user optimal principle when the link costs are constant.

The solution to the problem (25) - (27) is then given by

$$f_a^\ell = \sum_{i \in I} \sum_{k \in K_i} \delta_{ak} f_k \tag{31}$$

and the direction of descent is simply $d_a^\ell = (f_a^\ell - v_a^\ell)$.

The shortest paths calculations required in the solution of this subproblem may be carried out by any of the efficient methods currently available. (See for instance Gallo and Pallottino (1982)).

An iteration of the linear approximation algorithm is completed by finding the solution of

$$\underset{0 \le \lambda \le 1}{\text{Min}} \sum_a \int_0^{v_a^\ell + \lambda d_a^\ell} s_a(x)\, dx \tag{32}$$

or equivalently annuling its gradient, that is find λ^ℓ for which

$$\sum_{a \in A} s_a(s_a^\ell + \lambda^\ell d_a^\ell)\, d_a^\ell = 0 \tag{33}$$

For numerical reasons it is preferable to find λ by (33). The linearization is repeated for $\ell = 1,2,\ldots$ until a satisfactory solution is obtained.

This algorithm has several marked advantages. The necessary paths are generated as required and need not be kept from one iteration to the next. Thus the storage requirements are modest and do not increase with

the number of iterations. At each iteration only the arc flows v_a^ℓ need to be kept, in addition to the link costs $s_a(v_a^\ell)$ and the network. A drawback of this algorithm is that it converges slowly in the vicinity of the optimal solution. In practice this is not very limiting, since very accurate solutions are not needed for large scale problems.

The variable demand problem is just slightly more difficult to solve than the fixed demand problem. The linearized problem at iteration ℓ is

$$\text{Min} \sum_{i\in I} \sum_{k\in K_i} \sum_{a\in A} s_a(v_a^\ell)\delta a_k \cdot f_k - \sum_{i\in I} W_i(g_i^\ell)\, w_i \tag{34}$$

subject to

$$\sum_{k\in K_i} f_k - w_i = 0 , \quad i\in I \tag{35}$$

$$f_k \geq 0, \quad k\in K_i,\ i\in I,\ w_i \geq 0, \quad i\in I \tag{36}$$

As before, the objective (34) simplifies after some manipulations to

$$\text{Min} \sum_{i\in I} \left(\sum_{k\in K_i} s_k^\ell\, f_k - W_i(g_i^\ell) w_i \right) \tag{37}$$

which may be solved by determining u_i to be lengths of the shortest paths based on the current link cost $s_a(v_a^\ell)$ and determining the demand to be allocated to these shortest paths as follows, where g_i^{max} is an upper bound of g_i.

$$w_i^\ell = \begin{cases} g_i^{max} & \text{if } u_i < W_i(g_i^\ell) \\ g_i^\ell & \text{if } |u_i - W_i(g_i^\ell)| < \varepsilon \quad i\in I \\ 0 & \text{if } u_i > W_i(g_i^\ell) \end{cases} \tag{38}$$

The descent direction is $\{(f_a^\ell - v_a^\ell), (w_i^\ell - g_i^\ell)\}$ where f_a^ℓ is obtained by assigning the demand w_i^ℓ to the shortest paths. The step size is determined by finding λ^ℓ which annuls the gradient

$$\sum_{a\in A} s_a(v_a^\ell+\lambda(f_a^\ell-v_a^\ell))(f_a^\ell-v_a^\ell) - \sum_{i\in I} W_i(g_i^\ell+\lambda(w_i^\ell-g_i^\ell))(w_i^\ell-g_i^\ell) \qquad (39)$$

An interesting variant of the linear approximation method for solving the variable demand network equilibrium problem is that of partial linear approximation that was first suggested by S. Evans (1976) for solving the combined trip distribution/assignment problem. This problem is a variant of the variable demand problem where the demand functions are taken to be negative exponential functionsof the origin to destination travel costs,that is $G_i(u_i) \cong \exp(-\theta u_i)$, $i\in I$ and additional constraints are imposed on the total trips originating at origin p and terminating at destination q. This model may be stated as

$$\text{Min} \sum_{a\in A} \int_0^{v_a} s_a(x)dx + \theta \sum_{i\in I} g_i \ell n\, g_i \qquad (40)$$

subject to

$$\sum_{k\in K_i} h_k - g_i = 0, \quad i\in I \qquad (41)$$

$$\left.\begin{aligned} \sum_p g_{pq} &= D_q, \quad \forall q \\ \sum_q g_{pq} &= O_p, \quad \forall p \end{aligned}\right\} \; i=(p,q) \qquad (42)$$

$$h_k \geq 0, \quad k\in K_i, \; i\in I; \quad g_i \geq 0, \quad i\in I \quad . \qquad (43)$$

Evan's algorithm was interpreted by Stewart (1979) to be a partial linear approximation, where for the variable demand problem the user cost

functions are linearized, but the inverse demand functions are not. The resulting subproblem at iteration ℓ is

$$\text{Min} \sum_{i\in I} \sum_{k\in K_i} \sum_{a} s_a(v_a^{\ell})\delta a_k \cdot f_k - \sum_{i\in I} W_i(w_i) \tag{44}$$

subject to $$\sum_{k\in K_i} f_k - w_i = 0, \quad i\in I \tag{45}$$

$$f_k \geq 0, \quad k\in K_i, \quad i\in I; \quad w_i \geq 0, \quad i\in I \ . \tag{46}$$

This subproblem is still easy to solve by determining u_i to be the shortest paths based on the current link cost $s_a(v_a^{\ell})$ and then determining the demand to be allocated to these shortest paths as

$$w_i^{\ell} = G_i(u_i), \quad i\in I \ . \tag{47}$$

The partial linear approximation direction may be easily shown to be a descent direction (see Florian (1982)). Intuitively, this direction should be better, since the subproblem used for finding the direction of descent is "closer" to the original problem. Computational results (Frank (1978), LeBlanc and Farhangian (1981)) support this intuition. As this subproblem is not more difficult to solve than the linear approximation subproblems, it is attractive to use in practice.

The slow convergence of the linear approximation method in the vicinity of the optimal solution has stimulated interest in improving its rate of convergence without destroying its advantages. Holloway (1974) proposed an extended version of the method which requires the storage of all the extreme flows $\{f_a^{\ell}\}$ generated by the method. For $L>2$, the subproblem to be considered is the "restriction" of the problem, over the convex hull of these extreme points, that is

$$\left.\begin{aligned} &\text{Min} \sum_a S_a\Big(\sum_{\ell=1}^{L} \alpha_\ell f_a^\ell\Big) \\ &\text{subject to} \quad \sum_{\ell=1}^{L} \alpha_\ell = 1,\ \alpha_\ell \geq 0 \quad \ell = 1,2,\ldots,L \end{aligned}\right\} \tag{48}$$

where $S_a(\cdot) = \int_0^{(\cdot)} s_a(x)dx$, α_ℓ are the coefficients of each extreme point f_a^ℓ in the convex linear combination $v_a = \sum_{\ell=1}^{L} \alpha_\ell f_a^\ell$. For simplicity of the exposition, the fixed demand problem is reconsidered. The restricted subproblem (48) is solved, instead of carrying out the line search (33), either to optimality or to some prescribed ε-optimality. Then a new extreme point f_a^{L+1} is generated and the method proceeds as described above. The advantage of this approach is that a large variety of descent directions may be used on the restricted subproblem. Cantor and Gerla (1974), proposed the use of the projected gradient method to solve this subproblem. (Their work is probably independent of that of Holloway). Florian (1977) proposed the use of Wolfe's (1970) "away" step where directions are generated by computing

$$\underset{1\leq\ell\leq L-1}{\text{Max}} \sum_a s_a(v_a^L) f_a^\ell \tag{49}$$

and for ℓ^* which achieves this minimum the direction of descent is computed as

$$\left.\begin{aligned} &d_a = (\theta^*-1)(v_a^L - f_a^{\ell^*}),\ a\in A \\ &\text{where} \quad \theta^* = \frac{1}{1-\alpha_\ell^*} \end{aligned}\right\} \tag{50}$$

Alternatively, one may use the reduced gradient method to solve this

subproblem (see Florian, 1977) which would result in computing the reduced gradient

$$r_\ell = \sum_a s_a(v_a^L)(f_a^{\ell 1} - f_a^\ell) \quad , \quad \ell=1,2,\ldots,L \tag{51}$$

where $\ell 1$ is the index for the largest d_ℓ and the direction

$$d_\ell = \begin{cases} r_\ell & \text{if } r_\ell > 0 \\ \alpha_\ell r_\ell & \text{if } r_\ell < 0 \\ -\sum_{\ell \neq \ell 1} d_\ell & \text{if } \ell = \ell 1 \end{cases} \tag{52}$$

Computational results (Florian (1977), Guélat (1982)) indicate that the convergence rate is improved at the expense of storing f_a^ℓ, $\ell=1,\ldots,L$. A disadvantage of this approach is that experimentation is required to determine how many iterations to perform on the subproblem (48) before generating a new extreme point, and the number of regular iterations to perform before beginning to solve the restricted subproblem. The best choices may vary from one problem to another. These difficulties are partially overcome by using the parallel tangents method (PARTAN) credited to Shab, Buehler and Kempthorne (1964), that was developed for the minimization of a differentiable function of several variables. This method was applied by Boyce, Helgason and LeBlanc (1982) and Florian, Guélat and Spiess (1983). Essentially, it may be interpreted as carrying out exactly one step on the restricted problem (48) by using the direction generated by every other solution v^{L-2} and v^L

$$\left.\begin{array}{l} d_a^L = v_a^{L-2} + \theta(v_a^L - v_a^{L-2}), \quad a \in A \\ 0 \le \theta \le \theta^* \end{array}\right\} \qquad (53)$$

where θ^* is the maximum step size that maintains the nonnegativity of the weights α_ℓ in the convex linear combination $v_a = \sum_\ell \alpha_\ell f_a^\ell$, $a \in A$.

Although this direction is not guaranteed to be a descent direction, empirically it has yielded very good results. The implementation of this variant requires only the storage and update of α_ℓ, $\ell=1,\ldots,L$ but not of the extreme points f_a^ℓ; instead, the two most recent solutions v_a^{L-1} and v_a^{L-2} must be stored. Thus its additional storage requirements are less than those required by the straightforward application of a variant of Holloway's extended linear approximation method and looks at present as a most attractive method to use in practice.

Recently, Dembo (1982) suggested another variant of this algorithm which replaces the objective function by its quadratic approximation and a few steps of the linear approximation algorithm, or one of its variants, are taken on this objective function to generate a solution f_a^ℓ which serves to compute the direction of descent $d_a^\ell = (f_a^\ell - v_a^\ell)$. The combination of this approach with the PARTAN directions may result in an efficient variant of the linear approximation method which has desirable asymptotic convergence properties as well.

Other algorithmic approaches have been developed and tested computationally for this problem. Nguyen (1974) has adapted the convex simplex method by using a decomposition of the flows v_a by origin and keeping a

"copy" of the network with flows v_a^p for each origin p. The storage requirements of this method and the input and output manipulations required each time that an origin is considered, impede the application of the method, although empirically the computational results obtained were good. Another approach to solving this problem when the flows are decomposed by origin is to apply an algorithm based in finding negative cycles as proposed by Weintraub (1981). Bertsekas (1976) and Dembo and Klincewicz (1978) have proposed second-order algorithms for solving the problems for each origin. Conclusive computational comparisons are still to be performed.

When the number of paths for an origin/destination pair is relatively small, such as in an airline network or a railway network and the path information is necessary, it may be advantageous to decompose the problem by origin/destination pairs and cyclically solve smaller problems in the space of the path flows. The solution strategy used may be stated as follows.

Step 1. Given an initial solution, set j=0, m=0 and continue.

Step 2. If m=|I|, the total number of origin/destination pairs, terminate, otherwise set j=j modulo |I|+1 and continue.

Step 3. If the current flow is optimal for the j^{th} origin/destination pair, set m=m+1 and return to step 2; otherwise, solve the j^{th} subproblem, update the current flow, set m=1, and return to step 2.

Such solution approaches were suggested by Gibert (1968), Dafermos (1972), Leventhal, Nemhauser and Trotter (1973), Nguyen (1974) and Soumis (1978). The convergence of this decomposition approach is ensured by the

fact that the objective function is strictly convex and it may be shown to be strictly decreasing after each solution of a subproblem in Step 3. A rigorous proof may be found in Auslender (1976, pp. 94-96).

The subproblem to be solved for the origin/destination pair j is

$$\text{Min} \sum_a \int_0^{v_a^j + \sum_{i \neq j} \sum_{k \in K_i} \delta a_k \bar{h}_k} s_a(x)dx \tag{54}$$

subject to

$$\sum_{k \in K_j} h_k = \bar{g}_j \tag{55}$$

$$h_k \geq 0, \quad k \in K_j \tag{56}$$

and

$$v_a^j = \sum_{k \in K_j} \delta a_k \, h_k \, , \quad a \in A \; . \tag{57}$$

This problem bears a strong similarity to the restricted problem in Holloway's extended linear approximation method, since there is a single constraint on the path flows equation (55). Thus, one may apply any of the algorithms that have been mentioned previously to solve the restricted problem, such as the reduced gradient (Nguyen (1974), Florian and Nguyen (1974)), the projected gradient method (Soumis (1978)) or the convex simplex method (Dafermos (1972)). Alternatively, the subproblem may be converted into a circulation problem that may be solved with an algorithm which detects and resolves negative cycles (Ferland (1975)). As an example, the use of the reduced gradient method, combined with a restriction strategy would result in the following algorithm for a subproblem for an origin/destination pair.

Step 1. Compute the costs of the currently used paths $s_k, k\epsilon K_j^+$ where $k\epsilon K_j^+$, if $h_k > 0$. Find k_1 which corresponds to $h_{k_1} = \max_{k\epsilon K_j^+} h_k$. The reduced gradient is $r_k = s_{k_1} - s_k$, $k\epsilon K_j^+$.

The reduced gradient direction is

$$d_k = \begin{cases} (s_{k_1} - s_k) & \text{if } r_k > 0 \\ h_k(s_{k_1} - s_k) & \text{if } r_k < 0 \\ -\sum_{k\neq k} d_k & \text{of } k = k_1 \end{cases} \tag{58}$$

If an ε-solution has been obtained go to Step 4, otherwise, continue to

Step 2. Find the step size λ which solves

$$\text{Min} \sum_a \int_0^{v_a + \lambda f_a^j + Ha} s_a(x)dx \tag{59}$$

$$0 \leq \lambda \leq \min\left(\frac{-h_k}{d_k} \mid d_k < 0 \right)$$

where $f_a^j = \sum_{k\epsilon K_j^+} \delta a_k \, d_k$ and $Ha = \sum_{i\neq j} \sum_{k\epsilon K_i} \delta a_k \, \bar{h}_k$

Step 3. $h_k = h_k + \lambda d_k, \quad k\epsilon K_j^+$

$v_a^j = v_a^j + \lambda f_a^j + Ha, \quad a\epsilon A$

and return to Step 1.

Step 4. Compute the shortest path $\tilde{k}$ of cost $\tilde{s}_k = \min_{k\epsilon K_j} (s_k)$

if $\tilde{s}_k < \min_{k\epsilon K_j^+} (s_k)$, then the path k* is added to the set of paths $K_j^+ = K_j^+ \cup k^*$ and return to Step 1, otherwise terminate.

Evidently, depending on the algorithm chosen, the details of Steps 1 and 2 would change to reflect the nature of the descent direction chosen.

An interesting variant of the network equilibrium problem occurs when the paths are fixed and the problem of interest is to find the equilibrium flows with variable demand on the available paths. This problem was introduced by Petersen and Tulett (1982) in the context of rail networks, where the paths represent fixed train service. They suggest the use of a fixed point algorithm which is an alternative to the optimization approach.

It is appropriate to note here that the "system" optimal versions of the network equilibrium problem with fixed or variable demand, that is for instance the problem

$$\text{Min} \sum_{a \in A} v_a \cdot s_a(v_a) - \sum_{i \in I} \int_0^{g_i} W_i(y)dy) \tag{60}$$

subject to (2), (3) may be solved by any of the computational approaches mentioned above.

The network equilibrium model is a suitable representation of route choice behavior when congestion effects significantly increase travel costs. Since it is seldom true in practice that travelers have exact knowledge of the link times when congestion is not present, it is likely that more than one route will be used even if these are longer than the shortest. Dial (1971) proposed a route choice model based on the logit model which postulates that

$$h_{k'} = \bar{g}_i \exp(-\theta s_{k'}) / \sum_{k \in K_i} \exp(-\theta s_k) \tag{61}$$

subject to the usual flow conservation and nonnegativity constraints (23),

(24). This model has the equivalent optimization problem

$$\text{Min} \sum_{i \in I} \sum_{k \in K_i} h_k \, (\ell n \, h_k + \theta s_k) \tag{62}$$

The solution of this model by one of the algorithms described above requires the explicit storage of paths. Dial ingeniously circumvented this problem by restricting the paths to be considered and succeeded in solving this restricted version of the problem without requiring path enumeration. Although this model found much use in practice, the use of the logit path choice makes the resulting flows very sensitive to the network representation (see Burrell (1976)) and the restriction of the paths results in the model being unstable; slight modifications in the network may induce large changes in the resulting flows.

Fisk (1980) extended this model to the situation where congestion effects are present and demonstrated that the optimization problem

$$\text{Min } Z = \sum_{i \in I} \sum_{k \in K_i} h_k \, \ell n \, h_k + \sum_a \int_0^{v_a} s_a(x) dx \tag{63}$$

satisfies the route choice postulate (61) even when the link costs $s_a(v)$ are flow dependent. This may be easily verified by applying the Kuhn-Tucker conditions to (63), (2), (3). The solution of this model by an algorithm which does not require the explicit storage of the path flows h_k was suggested by Fisk (1980) and proved to converge by Powell and Sheffi (1980). The algorithm is remarkably simple: it follows the same steps as the linear approximation method except that the step size λ is predetermined to be $\frac{1}{L}$ where L is the current number of iterations. Although the convergence of this algorithm is ensured, it exhibits very slow empirical convergence

and it is also disadvantaged by the lack of a suitable stopping criterion.

Daganzo (1980) considers such problems in the space of costs rather than in the space of flows and suggests different solution approaches.

We turn our attention now to the solution of network equilibrium problems which do not have equivalent optimization problems, but may be formulated as nonlinear complementarity problems or variational inequality problems. Although these problems are more difficult to solve than optimization problems, recent developments have made it possible to tackle successfully certain classes of these problems and some practical applications have been achieved.

We reconsider not the variational inequality formulation of the network equilibrium problem in the form

$$s(v^*)^T(V-v^*) - W(g^*)^T(g-g^*) \geq 0 \ .$$

There are many variants of the network equilibrium problem which are captured by this formulation. When the flows may occur on several modes (car, bus, train) on some or all of the links of the network, the user costs for travel on a link a depends on the flow of all modes that use that link, that is

$$s(v) = (s_a^m(v),\ a \in A, \quad \forall\, m) \tag{64}$$

and the demand for travel on mode m depends on the vector of travel costs by all modes, that is

$$G(u) = (G_i^m(u),\ i \in I, \quad \forall\, m) \tag{65}$$

where u is the vector of travel costs for origin/destination pair i by all modes. If the vector function G(u) admits a smooth inverse W(g), the model is well formulated as the variational inequality (14).

A similar formulation results for situations where the total demand for travel by all modes is constant and the demand function predicts the proportion of trips that occur by each mode. When two modes are considered say private car and bus, then $g_i^1 + g_i^2 = \hat{g}_i$, $i \in I$ and the model may be formulated by eliminating the demand of one of the modes, say mode 2. In that case, Florian and Spiess (1981) showed that the resulting variational inequality is

$$s(v^*)^T(v-v^*) - W(g^1*)(g^1-g^1*) \geq 0 \qquad (66)$$

where $W(g^1)$ is the mapping of inverse mode choice functions, which are taken to be strictly decreasing functions of g^1, the demand for travel by mode 1.

This variational inequality may be derived from (12) by the appropriate substitutions and some algebraic manipulations.

Network equilibrium problems may be much larger than problems that may be solved by general purpose algorithms for nonlinear complementarity problems or variational inequality problems. Nevertheless, recent progress in this area has led to the successful solution of large problems by using methods that amount to solving a sequence of network equilibrium problems which have equivalent optimization formulations. The two most natural approaches are the adaptation of the Jacobi and Gauss-Seidel methods for solving systems of linear and nonlinear equations. These methods are quite intuitive and their use was suggested even before their convergence was supported by rigorous proofs.

In the Jacobi method, given a feasible solution $(v,g)^{\ell}$ at iteration ℓ, the off-diagonal elements of the user cost and inverse demand functions are fixed by using the current solution elements. This results in user cost function $s(v,v^{\ell})$ and inverse demand function $W(g,g^{\ell})$. The corresponding variational inequality

$$s(v^*,v^{\ell})^T(v-v^*) - W(g^*,g^{\ell})(g-g^*) \geq 0 \qquad (67)$$

has an equivalent optimization formulation and may be solved by any of the methods discussed above. Depending on the nature of the problem, this solution strategy may result in a decomposition by origin/destination pairs, by origin, by mode or by origin and mode. The fixing of the off-diagonal elements yields the smallest subproblem which has an equivalent optimization problem. $(v,g)^{\ell+1}$ is obtained by solving (67) by an appropriate algorithm and repeating the process of redefining the cost and demand functions.

The Gauss-Seidel method is very similar, with the exception that when the variational inequality decomposes into more than one subproblem, the most recent information is used; that is, given that the variational inequality decomposes into subproblems m,m=1,...,M or

$$(v,g)^{\ell} = \{(v,g)_1^{\ell}(v,g)_2^{\ell}\ldots,(v,g)_M^{\ell}\}$$

then $(v,g)_m^{\ell+1}$ is obtained for m=1,...,M by solving the decomposed variational inequality which depends on the vector

$$\{v,g_1^{\ell+1},\ldots,(v,g)_{m-1}^{\ell+1}, (v,g)_{m+1}^{\ell},\ldots,(v,g)_M^{\ell}\}$$

Thus the variational inequality solved as an equivalent optimization problem in the m^{th} component of the original vector functions, where for each subproblem m the off-diagonal elements are fixed with the solution of subproblems

1,...,m-1 at iteration $\ell+1$ and with the solutions of subproblems m+1 to M at iteration ℓ. An iteration is completed by solving M subproblems.

Another approach which has been suggested is a Newton type linearization scheme which determines the vectors $(v,g)_m^{\ell}$, $\ell=1,2...$ iteratively by obtaining $(v,g)^{\ell+1}$, from $(v,g)^{\ell}$ by replacing the cost functions with their first order Taylor approximation at $(v,g)^{\ell}$. $(v,g)^{\ell+1}$ is the solution of the affine cost problem

$$s_{\ell in}(v^*,v^{\ell})^T(v-v^*) - W_{\ell in}(g^*,g^{\ell})^T(g-g^*) \geq 0 \qquad (68)$$

when $s_{\ell in}(v^*,v^{\ell}) = s(v^{\ell}) + \nabla s(v^{\ell})(v-v^{\ell})$ is the first order approximation of $s(v)$ at v^{ℓ} and $W_{\ell in}(g,g^{\ell})$ is similarly defined. This approach requires the solution of a sequence of variational inequalities with affine costs. This problem may be converted into a nonlinear complimentarity problem (see Aashtiani and Magnanti (1982)), however the pivoting methods available for its solution are not effective for large scale problems. Alternatively, one could solve each linearized subproblem by an appropriate decomposition scheme, such as the Jacobi or Gauss-Seidel methods.

Another approach to solve the general network equilibrium problem includes projection methods (Dafermos (1981), Bertsekas and Gafni (1980)). Recently, Lawphonganpovich and Hearn (1982) suggested a projection algorithm which adapts the approach of Bertsekas and Gafni with a simplified decomposition scheme which uses the restriction approach (48), akin to the extended versions of the linear approximation method. Yet another method proposed by Nguyen and Dupuis (1981) is a cutting plane algorithm applied to a dual formulation of the variational inequality formulation of the network

equilibrium problem. This formulation of the problem is an optimization problem with a nondifferentiable objective (see Hearn (1982) and Hearn and Nguyen (1982)). The detailed presentation of these methods far exceeds the scope of this paper.

The recent interest in solving the general network equilibrium problem has stimulated the study of the convergence of the solution approaches mentioned above. The convergence of the Jacobi method, proposed in this context by Florian (1977) and Abdulaal and LeBlanc (1979), Hall, Van Vliet and Willumsen (1980) has been studied by Pang and Chan (1981) and Florian and Spiess (1982) by using ideas similar to those proposed by Ahn (1979) for showing the convergence of the Jacobi method for an economic equilibrium model. Josephy (1979) and Pang and Chan have established local convergence criteria for Newton type linearization algorithms. Dafermos (1982a) has first studied global convergence properties of the Jacobi method and then Dafermos (1982b), has provided a unified way for proving the convergence of the Jacobi, Gauss-Siedel and projection methods for variational inequalities. Soumis (1978) and Pang and Chan (1982) have provided proofs for the convergence of the Gauss-Seidel method. The essence of most of these convergence results, which are obtained by showing that the algorithmic maps are contractive, is that the strong monotonicity of the cost functions, which ensures the uniqueness of the solution, is a necessary conditions for convergence. Furthermore, the Jacobian matrices of the cost functions must satisfy additional conditions for which the diagonal dominance (or block diagonal dominance) of these matrices is a sufficient condition. To date, the Jacobi method (see Florian et al. (1979)) and the Gauss-Seidel method (see Soumis (1978)) are the only methods that were applied in practice for solving general network equilibrium

problems. However, the rapid evolution of solution approaches in this field may lead to the emergence of other, more efficient methods.

3. ESTIMATING AN ORIGIN/DESTINATION MATRIX

In the presentation of the fixed demand network equilibrium model, it was assumed that $\bar{g}_i$, $i \in I$ are known. Actually, one of the most difficult problems in practice is the estimation of these demands, either by using econometric methods or a certain class of nonlinear networks models, known as trip distribution or spatial interaction models. In this section of the paper we consider this class of models, which may be considered as a certain class of transportation problems with convex costs.

The classical model used to determine an origin/destination matrix is the gravity or entropy model which is based on the assumption that the total by origins, O_p, $\forall$ p and by destination D_q, $\forall$ q are known and the matrix elements g_{pq}, $\forall$ $(p,q) \equiv i$ must satisfy the constraints

$$\sum_q g_{pq} = O_p, \quad \forall\ p \tag{69}$$

$$\sum_p g_{pq} = D_q, \quad \forall\ q \tag{70}$$

$$g_{pq} \geq 0, \quad \forall\ (p,q) \tag{71}$$

where $\sum_p O_p$ is assumed equal to $\sum_q D_q$.

In absence of other information it is postulated that the origin/destination matrix to be estimated is the "most likely to occur", which leads to the optimization problem

$$\text{Max} - \sum_p \sum_q g_{pq} \ln g_{pq} \tag{72}$$

subject to (69)-(71). The objective has the interpretation of entropy maximization and the formalism stems from information theory (see Jaynes (1957)). The model was introduced to transportation and regional analysis by Wilson (1967, 1970). When some a priori information is known about the matrix, say g^o_{pq}, $\forall$ (pq), Kullback (1959) and Snickars and Weibull (1977) have suggested the use of the objective

$$\text{Max} - \sum_p \sum_q g_{pq} \ln \left(\frac{g_{pq}}{g^o_{pq}}\right) . \tag{73}$$

Yet another popular variant of this model is the introduction of a constraint on the total travel time

$$\sum_p \sum_q g_{pq} u_{pq} = U \tag{74}$$

wuich leads to the objective function

$$\text{Min} \sum_p \sum_q g_{pq} (\ln g_{pq} + \theta u_{pq}) \tag{75}$$

where θ has the interpretation of the dual variable associated with the constraint (74). It is trivial to verify, by applying the Kuhn-Tucker conditions that any solution of (75) has the general form

$$\begin{aligned} g_{pq} &= \exp(\alpha_p + \beta_q - 1) \exp(-\theta u_{pq}) \quad \forall (pq) \\ &= r_p \cdot s_q \cdot \exp(-\theta u_{pq}) \end{aligned} \tag{76}$$

where α_p and β_q are the dual variables associated with the corresponding constraints in (69) and (70). Since (75) is a convex function, with the

use of the convention $g_{pq} \ell n\, g_{pq} = 0$ when $g_{pq} = 0$, it is possible to obtain solutions to this class of problems by using any primal convex programming algorithm. However, the fact that the primal variables may be expressed in terms of the dual variables, makes it possible to obtain very efficiently solutions to this model by using a primal-dual method, commonly refered to as the balancing method. It consists of the following steps.

Step 1 $\quad \ell = 0,\ g_{pq}^{o} = \exp(-\theta u_{pq}),\ \forall\,(pq)$

$$r_p^o = 1,\ \forall\, p\ ;\quad s_q^o = 1,\ \forall\, q$$

Step ℓ $\quad (\ell = 1,2,\ldots)$

(a) $$r_p^{\ell+1} = \frac{O_p}{\sum_q g_{pq}^{\ell}} \cdot r_p^{\ell}\ ;\quad g_{pq}^{\ell+\frac{1}{2}} = \frac{O_p}{\sum_{q'} g_{pq'}^{\ell}} \cdot g_{pq}^{\ell},\ \forall\, p,q$$

$\forall\, p$

(b) $$s_q^{\ell+1} = \frac{D_q}{\sum_p g_{pq}^{\ell+\frac{1}{2}}} \cdot s_q^{\ell}\ ;\quad g_{pq}^{\ell+1} = \frac{D_q}{\sum_{p'} g_{p'q}^{\ell+\frac{1}{2}}}\ g_{pq}^{\ell+\frac{1}{2}},\ \forall\, q,p$$

$\forall\, q$

The algorithm is terminated when $\|r_p^{\ell+1} - r_p^{\ell}\| \le \varepsilon$ and $\|s_q^{\ell+1} - s_q^{\ell}\| \le \varepsilon$. It is interesting to note that $\alpha_p = \ell n\, r_p$ and the difference between two successive values of the dual variables $(\alpha_p^{\ell+1} - \alpha_p^{\ell}) = \ell n\, (\frac{O_p}{\sum_q g_{pq}^{\ell}})$.

This method dates back to at least 1937, when Kruithof used it for the prediction of telephone traffic distribution. Deming and Stephan (1940) independently rediscovered the method and applied it to a cross-classification problem in statistics as a simplification to least squares fitting. A proof of convergence is easily obtained for the version of the problem when all $u_{ij} < \infty$. In that case, the Lagrangian dual of the problem (75) subject to (69)-(71) is

$$\max_{\alpha,\beta} \sum_p \alpha_p O_p + \sum_q \beta_q D_q - \sum_p \sum_q \exp(\alpha_p + \beta_q - 1 - \theta u_{pq}) \qquad (77)$$

and the balancing method is simply the application of the coordinate ascent method to each row and column until convergence is achieved.

This model has other interesting properties. For instance $g_{pq} > 0$, $\forall (p,q)$, if $u_{pq} < \infty$ and $\theta > 0$; otherwise, if $g_{pq} = 0$ then it follows from (76) that either $\alpha_p(\beta_q) = \infty$ which would violate (69) or (70) since if $\alpha_p(\beta_q) = 0$ then $\sum_q g_{pq} \left(\sum_p g_{pq}\right) = 0$.

Another interesting variant of this model is when the double constraints (69), (70) are replaced by inequalities. For instance, when

$$\sum_p g_{pq} \leq O_p. \quad \forall p \qquad (78)$$

replaces (69), the solution may be found by modifying the balancing method by replacing the balancing factor in Step $\ell(a)$ to be

$$\min\left(1, \frac{O_p}{\sum_q g^{\ell}_{pq}}\right) \quad \text{instead of} \quad \left(\frac{O_p}{\sum_q g^{\ell}_{pq}}\right)$$

This variant of the model was studied by Jefferson and Scott (1979). Lebeuf, Jörnsten and Stewart (1980) showed that the balancing method and its variants may be viewed as special cases of Bregman's (1967) non orthogonal projection method to solve certain problems of convex programming. Variants of the balancing method have been studied by Murchland (1977), Robillard and Stewart (1974), Evans and Kirby (1974). Anderson (1979) studied the convergence of the balancing method and other iterative methods based on simplified Newton iterations by using conjugate duality theory.

An alternative model that may be used to estimate an origin/destination matrix is the minimization of the sum of squares $\sum_p \sum_q (g_{pq}-g^o_{pq})^2$ where g^o_{pq} are observations. This results in the minimization of a quadratic function

$$\text{Min} \sum_p \sum_q (a_{pq} + b_{pq}\, g_{pq})\, g_{pq} \tag{79}$$

subject to (69)-(71). Although this problem may be solved by any algorithm for quadratic programming, the special structure of the constraints make it possible to apply a dual algorithm, such as done by Okuchi and Kaji (1981). In this case, it is again possible to find a closed form solution of the primal variables as function of the dual variables which is

$$g_{pq} = \max \left\{ 0,\ \frac{\alpha_p+\beta_q-a_{pq}}{2b_{pq}} \right\} \tag{80}$$

Thus the problem may be solved by a coordinate ascent method to find the optimal dual variables. However, since the equation (80) is not multiplicative, the value of the primal variables must be computed when the dual solution is known. Although computational comparisons with other methods to solve quadratic cost transportation problems have not been carried out, this approach is worthy of serious consideration.

Another approach to the estimation of an origin/destination matrix is more closely related to the network equilibrium problem. Instead of assuming that the row and column totals are known, it is assumed that observations on the arc flows $\hat{v}_a$, $a \in A$ are available and the traffic satisfies the equilibrium route choice hypothesis. Is it possible then to find an

origin/destination matrix which would reproduce the link flows? The answer is affirmative; moreover there is an infinity of matrices which satisfy this requirement. This question was studied by Nguyen (1977) who stated the problem as a network equilibrium problem with variable demand, where $\hat{u}_i$ are the lengths of the shortest paths computed with the link costs $s_a(\hat{v}_a)$, $a \in A$

$$\text{Min} \sum_a \int_0^{v_a} s_a(x)dx - \sum_{i \in I} \hat{u}_i \, g_i \tag{81}$$

subject to
$$\sum_{k \in K_i} h_k - g_i = 0, \quad i \in I$$
$$h_k \geq 0, \quad k \in K_i, \quad i \in I$$
$$g_i \geq 0, \quad i \in I \; .$$

The objective of (81) is strictly convex in $\{v_a\}$, hence v_a^*, $a \in A$ are unique. Any solution of (81) has the property that $u_i^* = \hat{u}_i$ for $g_i^* > 0$ and $v_a^* = \hat{v}_a$, $a \in A$. Any solution g_i^*, $i \in I$ is an origin/destination matrix which reproduces the observed flows. In spite of the elegance of this result, the nonuniqueness of g_i^*, $i \in I$ represents a significant drawback. An alternative way to consider the problem is to find the most likely matrix, that of maximum entropy, among those that reproduce the observed flows. This problem (see Jörnsten and Nguyen (1979) and Nguyen (1981)) may be formulated as follows.

$$\text{Min} \sum_{i \in I} g_i \, \ell n \, g_i \tag{82}$$

subject to
$$\sum_{i \in I} g_i = G \tag{83}$$

where G is a prescribed total number of trips and $g \in \Omega$, where Ω is the set

of g which are the solution of (81). This problem may be formulated as well in the direct form of minimizing (82) subject to (83), conservation of flow and nonnegativity constraints and subject to

$$\sum_a s_a(v_a)(\tilde{v}_a - v_a) - \sum_{i \in I} \hat{u}_i \, (\tilde{g}_i - g_i) \geq 0 \qquad (84)$$

where $\tilde{v}_a$, $\tilde{g}_i$ are any other possible solution.

Nguyen (1981) proposed a solution method for this rather difficult problem, which may characterized as a cutting plane method which resorts to a sequence of fixed demand network equilibrium computations and balancing problems.

Finally, another approach which is worth mentioning is the solution of the combined trip/distribution assignment problem (40)-(43) which may be fit to observed data by calibrating θ to reproduce the sum of link integrals part of the objective function, as suggested by Erlander, Nguyen and Stewart (1979).

Before concluding this section it is worthwhile to comment on the use of gravity models in urban and regional planning. The model (75) subject to (70), (71) is referred to as a residential location model since the column totals or attractions are fixed, which usually represent the employment and service activities. The resulting row sums $\sum_q g^*_{pq} = O^*_p$ are taken as predictions of the residential location choices mode. Hutchinson (1982) gives a good overview of such applications. The combined trip/ distribution assignment model has also been used in such applications, with the aim of taking into account the effects of congestion effects on

on urban location patterns. Los (1980) and Boyce (1980) describe various ways of adapting such models for this purpose. Florian and Nguyen (1978) studied the properties of combined trip distribution/assignment models of two or more modes that is pertinent in such applications.

4. THE SPATIAL PRICE EQUILIBRIUM PROBLEM

Samuelson (1952) first formulated the partial spatial price equilibrium model of interregional (or international trade) of one commodity, first stated by Cournot (1838) as a mathematical program by introducing an artificial objective function ("net-quasi welfare function" or "net social cost function") and pointed out that the transportation problem of linear programming is a special case of this formulation. Takayama and Judge (1964, 1971) later extended this model by considering multiple commodities, multiple time periods and showed that there is a dual formulation of the problem where the prices, rather than the interregional flows, are the decision variables. They proposed solution algorithms based on quadratic programming formulations (see also Hartwick (1971)) that result from the assumption of linear demand and supply functions in each region.

The spatial price equilibrium model of one commodity determines an interregional trade equilibrium in which demand prices equal to the sum of supply costs and transportation costs, and in which there is price efficiency: a trade equilibrium is not established as consumers in a given region pay different prices for the same product from different regions and as long as there are supply sources with lower total prices than those actually paid. If these conditions were not satisfied, consumer would

rearrange their demands in favour of the supply regions with the cheapest total prices.

The single commodity model is formulated by considering n regions which both supply and consume this commodity. Each region is a market where the price p_i for supplying the quantity d_i is given by a supply function $S_i(d_i)$, i=1,...,n and the demand price Π_j for buying quantity b_j is given by $W_j(b_j)$, j=1,...,n, the inverse of the demand function $G_j(\Pi_j)$. It is assumed that $S_i(d_i)$ is a continuous monotone function of d_i, $d_i \geq 0$, and that $G_j(\Pi_j)$ is a continuous bounded antitone function of b_j, $b_j \geq 0$. Interregional shipments t_{ij}, i,j=1,...,n if the product are possible and cost c_{ij} per unit. A spatial price equilibrium is established when the demand price equals to the supply price and the transportation cost for all positive interregional flows; if the demand price is less than the supply price and the transportation cost, then no interregional flow occurs. These equilibrium conditions may be stated as

$$S_i(d_i) + c_{ij} - I_j(b_j) \begin{cases} = 0 & \text{if } t_{ij} > 0 \\ \geq 0 & \text{if } t_{ij} = 0 \end{cases} \qquad (85)$$

i=j is permitted and c_{ij} need not be zero. In addition, the interregional flows and the equilibrium values of the amounts supplied and demanded satisfy conservation of flow

$$d_i = \sum_j t_{ij}, \quad \forall\, i \quad ; \quad b_j = \sum_i t_{ij}, \quad \forall\, j \qquad (86)$$

and nonnegativity $\quad t_{ij} \geq 0 \quad \forall\, (ij) \qquad (87)$

Thus, the problem may be viewed as a generalization of the transportation problem. As shown by Samuelson (1952) and then by Takayama and Judge (1964), the model has an equivalent convex cost optimization formulation which consists of the objective

$$\text{Min} \sum_i \int_0^{d_i} S_i(x)dx + \sum_i \sum_j c_{ij}\, t_{ij} - \sum_j \int_0^{b_j} I_j(y)dy \qquad (88)$$

It is straightforward to verify that the Kuhn-Tucker conditions for (88) subject to (86), (87) are equivalent to (85). The resemblance of this model to the network equilibrium problem is remarkable. In fact, Samuelson's work probably inspired that of Beckmann et al. (1956).

Most of the applications of this model (see for instance Judge and Takayama (1973)) have been achieved for specialized versions of this model when the functions $S_i(d_i)$ and $I_j(b_j)$ are linear, and by consequence, the objective (87) is a quadratic function, thus making it possible to apply standard methods for quadratic programming. Glassy (1978) suggests a dual approach for this version of the problem. Many empirical applications of this model have been made in its quadratic programming formulation, in particular in the field of agricultural economics. Schmitz and Bawden (1973) studied the international trade of wheat and Nagy et al. (1980) analyzed the impact of changes in freight rates on Canadian wheat exports. Kennedy (1974) applies the model to the determination of the state of the world oil market.

In fact, the solution of this model with nonlinear supply and cost functions is not much more difficult than the solution of the model with quadratic cost functions. Rowse (1981) used the nonlinear programming code MINOS (see Murtagh and Saunders (1977)) to solve a dual formulation of this

problem. When the problems considered are not very large, the use of a standard very efficient code such as MINOS is well warranted. For large problems, such as those considered by Delorme and Rousseau (1982), in the context of supply and demand of health care services, other solution approaches must be devised, based on one of the primal methods for minimizating a convex function subject to linear constraints. Poleto et al. (1980) apply Benders decomposition for the solution of spatial price equilibrium models.

The development of an algorithmic approach for the case when the cost functions are not quadratic is facilitated by a property of this model when the unit transportation costs are constant. Florian, Ferland and Zubieta (1982) showed that the set $T = \{(i,j) | t^*_{ij} > 0\}$ is an optimal solution of (88) subject to (85), (86) is a tree or a forest. This may be easily demonstrated by supposing that the optimal quantities supplied and demand d^*_i, $\forall$ i, b^*_j, $\forall$ j are known. Then the t^*_{ij} may be obtained by solving the transportation problem Min $\sum_i \sum_j c_j t_{ij}$ subject to (86), (87) with $d_i = d^*_i$, $\forall$ i and $b_i = b^*_i$. It follows that the set T is a tree,or a forest if the solution is degenerate. This property suggests the use of algorithms which identify a set of basic variables such as the reduced gradient or the convex simplex method.

When the supply functions depend on all the quantities supplied, that is $p_i = S_i(d)$, $\forall$ i and the inverse demand functions depend on all the quantities bought, $b_j = G_j(\Pi)$, the equilibrium conditions (85) are no longer equivalent to the optimization problem (88) unless the Jacobian of the vector function S(d) and G(Π) are symmetric. In that case, the spatial price equilibrium problem may be stated as a variational inequality or a nonlinear complementarity problem, in a manner which is entirely analogous

to the analysis of the network equilibrium problem. The variational inequality formulation is

$$\sum_i S_i(d^*)(d_i-d_i^*) + \sum_i \sum_j c_{ij}(b_{ij}-b_{ij}^*) - \sum_j W_j(b^*)(b_j-b_j^*) \geq 0 \quad (89)$$

subject to (86), (87). The problem may be solved by the adaptation of any of the methods which are used to solve the variational inequality formulation of the network equilibrium problem. When the cost functions are affine, Asmuth, Eaves and Peterson (1979) suggested the application of Lemke's algorithm to solve this problem since in that case, its nonlinear complementarity formulation reduced to a linear complementarity problem. Pang and Lee (1981) suggest a parametric linear complementarity technique for solving such problems. To date, computational experience with such approaches is scarce.

Several extensions of this model are readily available. In many instances of real-world transportation problems, the cost of shipping one unit of product for a given region pair is a nonlinear function $c_{ij}(t_{ij})$ of the quantity transported b_{ij}. The objective (88) may be modified to reflect nonlinear transportation costs by replacing the term $\sum_i \sum_j c_{ij}\ t_{ij}$ by $\sum_i \sum_j \int_0^{t_{ij}} c_{ij}(z)dz$. It is reasonable to model costs in this way when the transportation market under consideration is segmented into as many markets as there are region pairs, and where each submarket is captive to one transportation firm, which follows a price policy reflected by the function $c_{ij}(t_{ij})$. However, if this function is decreasing, the objective (88) is no longer necessarily a convex function.

MacKinnon (1975) suggests a slightly simpler, but perhaps more interesting extension. Suppose that a single firm controls the transportation

market and the price of a unit of transportation is a function of the total volume of transportation services $\sum_i \sum_j \tau_{ij}\, t_{ij}$, where τ_{ij} is the amount of transportation service required to ship one unit of commodity between i and j. Let $C(\sum_i \sum_j \tau_{ij} t_{ij})$ be the price for one unit of transportation service where $C(\cdot)$ is a continuous monotone increasing function. The equilibrium conditions (85) may be rewritten as

$$S_i(d_i) + \tau_{ij} C(\sum_i \sum_j \tau_{ij} t_{ij}) - W_j(b_j) \begin{cases} = 0 \text{ if } t_{ij} > 0 \\ \geq 0 \text{ if } t_{ij} = 0 \end{cases} \tag{90}$$

subject to (86), (87). Again there is an equivalent optimization problem which is

$$\text{Min} \sum_i \int_0^{d_i} S_i(x)dx + \int_0^{\sum_i \sum_j \tau_{ij} t_{ij}} C(z)dz - \sum_j \int_0^{b_j} W_j(y)dy \tag{91}$$

subject to the same constraints.

The spatial price equilibrium problem may be generalized to consider multiple products. Each region may be considered as a market where the price p_i^m for supplying quantity d_i^m of product m is given by a supply function $S_i^m(d_i)$ where $d_i (= d_i^m, m=1,\ldots,M) \geq 0$, is the vector of all the products offered. It is assumed that $S_i(d_i)$ is a continuous function of d_i and its restriction with respect to all products, except one, say $d_i^{m'}$, is a monotone increasing function of $d_i^{m'}$. The demand b_j^m, $b_j^m \geq 0$, for product m in market j, is given by a demand function $G_j(\pi_j)$ where $\pi_j (= \pi_j^m, m=1,\ldots,M)$ is the vector of demand prices. It is assumed that $G_j^m(\pi_j)$ is a continuous function of π_j and its restrictions with respect to all prices except one, say $\pi_j^{m'}$ is a monotone

upper bounded decreasing function of $\pi_j^{m'}$. It is also assumed that the mapping $G_j(\pi_j)$ admits a smooth inverse $W_j(b_j)$. For generality, it is also assumed that interregional flows may use a general network composed of links $a, a \in A$. The cost function $c_a^m(v_a)$, $\forall$ m, $\forall$ a is the cost of shipping commodities m over link a where $v_a (= v_a^m, m=1,\ldots,M)$ is the vector of commodity link flows. Again, $c_a(v_a)$ is assumed to be a continuous monotone function. Let h_k, $k \in K_{ij}^m$ be the flow on path k between region pair (ij) of commodity m; K_{ij}^m is the set of paths available for commodity m and region pairs (i,j). The cost of path k is $c_k(v) = \sum_a \delta_{ak} \cdot c_a^m(v_a)$, $k \in K_{ij}^m$, $\forall$ i,j,m. The path flows satisfy conservation of flow

$$t_{ij}^m - \sum_{k \in K_{ij}^m} h_k = 0, \quad \forall\ i,j,m \tag{92}$$

and

$$d_i^m = \sum_j t_{ij}^m, \quad \forall\ i,m \ ; \quad b_j^m = \sum_i t_{ij}^m \quad \forall\ j,m \tag{93}$$

$$h_k \geq 0, \quad k \in K_{ij}^m, \quad \forall\ i,j,m \tag{94}$$

and the equilibrium conditions are

$$S_i^m(d_i^*) + c_k(v^*) - W_j^m(b_j^*) \begin{cases} = 0 & \text{if } h_k^* > 0 \\ \geq 0 & \text{if } h_k^* = 0 \end{cases} \quad k \in K_{ij}^m, \ \forall\ i,j,m \tag{95}$$

This version of the model may be converted as well into a variational inequality formulation by using the same methodology as that used in analyzing the network equilibrium problem. The result is

$$\sum_i \sum_m S_i^m(d_i^*)(d_i^m - d_i^{m*}) + \sum_a \sum_m c_a^m(v_a^*)(v_a^m - v_a^{m*}) - \sum_m \sum_j W_j^m(b_j^*)(b_j^m - b_j^{m*}) \geq 0 \tag{96}$$

Under the appropriate symmetry conditions on the Jacobians of the supply, demand and link cost functions, the problem has an equivalent convex cost minimization problem. Barros et al. (1977) use such a formulation to study the milk products market in Chile. Hogan (1975) describes the PIES equilibrium for oil products which has a similar structure to this model, except that the supply sector is modeled by a linear program.

The multiproducts spatial price equilibrium problem may be solved by using any of the approaches that are used for the network equilibrium problem since the two problems have similar structures. Barros et al. (1977) suggest a decomposition by commodity. Hogan (1975) employed the Jacobi method to solve the PIES model; Pang (1981) used a decomposition by product coupled with an overrelaxation algorithm to solve affine cost multiproduct models. Florian and Los (1982) give a variety of approaches for constructing solution algorithms for this problem, both when an equivalent optimization problem exists and when the variational inequality (96) must be solved. Although the spatial price equilibrium problem and the network equilibrium are very similar and as Magnanti (1980) showed, they may both be viewed as special cases of a more general linear activity equilibrium model, they are sufficiently different to merit individual attention. For instance, the linear approximation method, which is rather efficient in solving network equilibrium problems, has not been nearly as useful for solving single commodity spatial equilibrium problems (see Florian, Ferland and Zubieta (1982)). To date, most of the applications in practice were achieved with linear supply and demand functions. The algorithmic approaches known at present may open the way for nonlinear model applications, if such supply and demand functions may be successfully calibrated.

5. CONCAVE COST NETWORK FLOW TRANSPORTATION MODELS

In contrast to models which involve the minimization of convex cost function over a network, that may be solved by a variety of algorithmic approaches, concave cost network flow models are much more difficult to solve. In the transportation context, such problems occur when there are fixed costs associated with a location decision or when the link costs exhibit economies of scale. Certain versions of network design models, particularly in the field of telecommunications, see for instance Yaged (1971), Zadeh (1973, 1974), naturally give rise to concave cost network flows models of single or multiple commodities. In this section only those problems which have a specific transportation application are addressed. These problems form but a small subset of concave cost network flow problems. Effectively this choice limits the discussion to transportation or transhipment problems with concave costs.

A single commodity transhipment model with concave cost functions is formulated by the normative model

$$\text{Min} \sum_{(x,y)\in A} s_{xy}(v_{xy}) \tag{97}$$

$$\text{subject to} \quad \sum_y v_{xy} - \sum_z v_{zx} = \begin{cases} \sum_q g_{pq} & x = p \\ 0 & x \neq p,q \\ -g_{pq} & x = q \in Q \end{cases} \tag{98}$$

$$\ell_{xy} \leq v_{xy} \leq c_{xy}, \quad \forall (x,y) \tag{99}$$

where flow form a single destination p is destined for nodes $q \in Q$, the arc

flows are subject to capacity constraints c_{xy}, lower bounds ℓ_{xy} and the cost functions $s_{xy}(v_{xy})$ are concave, but not necessarily continuous or differentiable.

This model captures as special cases location problems, where certain arcs may represent production or storage facilities to be located, and the flows on these arcs represent implicitly the decisions to open or locate a facility on that arc, Such problems occur in the shipment of agricultural products, such as considered and applied by Rust and Bornstein (1982).

Since the concave objective is minimized at an extreme point of the constraints (98), (99) and every extreme point is a local minimum, the solution of such problems in practice have been achieved by the application of an implicit enumeration algorithm, due to Resch and Barton (1970) and specialized for location problems by Soland (1974). It is based on linear approximations of the objective function (97) obtained by replacing the concave link cost functions with the linear approximation.

$$s_{xy}^{lin} = \frac{s_{xy}(c_{xy}) - s_{xy}(\ell_{xy})}{c_{xy} - \ell_{xy}} \qquad (100)$$

and solving a linear cost network flow problem. The resulting solution v_{xy}^{lin} is used to refine the linear approximation by bisecting each arc for which $0 < v_{xy}^{lin} < c_{xy}$ at v_{xy}^{lin} and by generating two descendent approximate problems: one in which the capacity on the link becomes v_{xy}^{lin} and one on which the lower bound becomes v_{xy}^{lin}. The solution of each of these problems provides lower bounds on the objective function and a new solution which

may serve again to continue the bisection which determines the branching in the implicit enumeration scheme. The linear approximation (100) may be adapted for concave cost functions which are not continuous and various branching stragegies may be used. In practice, this algorithm,although rather simple, has proved very effective as evidenced by the computational results reported in the above mentioned papers. LeBlanc (1976) applied this approach to a rail network design model.

The theoretical approaches based on the characterization of an extreme flow in a network have not led to practicable solution methods. As is well known, see Roy (1970), Florian, Rossin Arthiat and De Werra (1971), an extreme flow has the property that each loop which carries positive flow must have at least one saturated arc. Except for very special structure network, this result is not helpful in designing solution approaches. On the other hand, when the network is uncapacitated, an extreme flow is a tree (or a forest) (see Zangwill (1968)). This property was exploited by Gallo and Sodini (1978) to obtain efficiently a locally optimal solution. They define a locally optimal solution to be an extreme that has a lower cost than any of its adjacent extreme flows and show that this local optimality may be tested by the application of a shortest path algorithm. Gallo and Sodini (1978) extended this approach to multiple commodity problems in uncapacitated networks. Gallo, Sandi and Sodini (1980) derive an enumerative approach based on a similar approach coupled with linear approximations of the arc cost functions to obtain lower bounds.

Another approach to solve uncapacitated concave cost flow problems is by applying dynamic programming, such as done by Zangwill (1968) and more recently by Erickson, Monma and Veinott (1981). Since the problem is NP-hard even for very special planar networks, the resulting algorithms have exponential time bounds. Although interesting from a theoretical point of view, it is doubtful whether they are of interest for practical applications, and the branch and bound approach seems preferable. The inherent difficulty of this class of network flow problems continues to generate interest in its efficient solution.

REFERENCES

AASHTIANI, H. and MAGNANTI, T., "A Linearization and Decomposition Algorithm for Computing Urban Traffic Equilibrium". Proceedings of the 1982 IEEE Large Scale Systems Symposium (Oct. 1982).

AASHTIANI, H. and MAGNANTI, T., "Equilibria on a Congested Transportation Network". SIAM Journal on Algebraic and Discrete Methods 2, 213-226 (1981).

ABDULAAL, M. and LE BLANC, L., "Methods for Combining Model Split Equilibrium Assignment Models". Transportation Science 13, 292-314 (1979).

AHN, B., Computation of Market Equilibria for Policy Analysis: The Project Independence Evaluation System (PIES) Approach. Garland Publishing Inc., New York, NY (1979).

AHN, B. and HOGAN, W., "On Convergence of the PIES Algorithm for Computing Equilibria". Operations Research 330, 281-300 (1982).

ANDERSSON, P.A., "On the Convergence of Iterative Methods for the Distribution Balancing Problem", Report Lith-Mat-R-79-14, Dept. of Mathematics, Linkoping Institute of Technology (1979).

ASMUTH, R., "Traffic Network Equilibrium". Technical Report SOL-78-2, Stanford University, Stanford, CA (1978).

ASMUTH, R. EAVES, B.C. and PETERSON, E.L., "Computing Economic Equilibria on Affine Networks with Lemke's Algorithm". Math. Operations Res. 4, 209-214 (1979).

AUSLENDER, A., Optimisation, méthodes numériques, Masson (1976).

BARROS, O., GOMEZ, A. and WEINTRAUB, A., "Distribution Equilibrium Problems with Classic Supply and Demands" Publication #77/10/C, Departamento de Industrias, Universidad de Chile, Santiago (1977).

BECKMANN, M.J., McGUIRE, C.B. and WINSTEN, C.B., Studies in the Economics of Transportation. Yale University Press, New Haven, CT (1956).

BERTSEKAS, D., "Algorithms for Optimal Routing of Flow in Networks", Coordinated Science Lab. Working Paper, University of Illinois at Urbana-Champaign (1976).

BERTSEKAS, D. and GAFNI, E., "Projection Methods for Variational Inequalities with Application to the Traffic Assign-

ment Problem". Technical Report, Laboratory for Information and Decision Systems, MIT, Cambridge, MA (1980).

BOYCE, D., "A Framework for Constructing Network Equilibrium Models of Urban Location" Transportation Science 14, 71-96 (1980).

BREGMAN, L.M., "The Relaxation Method for Finding the Common Point of Convex Sets and its Application to the Solution of Problems in Convex Programming". USSR Comp. Math. Math. Phys. 7, 3, 200-217 (1967).

BRUYNOOGHE, A., GIBERT, A. and SAKAROVITCH, M., "Une méthode d'affectation du trafic". In Fourth Symposium on the Theory of Traffic Flow, Karlsruhe (1968).

BURREL, J.E., "Multiple Route Assignment: A Comparison of Two Methods" in Traffic Equilibrium Methods, Proceedings 1974, edited by M. Florian, Springer Verlag Lecture Notes in Economics and Mathematical System 118 (1976).

CANTOR, D. and GERLA, M., "Optimal Routing in a Packet Switched Computer Network", IEEE Transactions on Computers 10, 1062-1068 (1974).

COURNOT, A.A., Mathematical Principles of the Theory of Wealth, Ch. X (1838).

DAFERMOS, S., "An Extended Traffic Assignment Model with Applications to Two-Way Traffic". Transportation Science 4, 336-389 (1971).

DAFERMOS, S., "An Iterative Scheme for Variational Inequalities". Mathematical Programming to appear (1982).

DAFERMOS, S., "Relaxation Algorithms for the General Asymmetric Traffic Problem". Transportation Science 16 (1982).

DAFERMOS, S., "The Traffic Assignment Problem for Multiclass-User Transportation Networks". Transportation Science 6, 73-87 (1972).

DAFERMOS, S., "Traffic Equilibrium and Variational Inequalities". Transportation Science 14, 42-54 (1980).

DAFERMOS, S. and NAGURNEY, A., "Sensitivity Analysis for the General Asymmetric Network Equiblibrium Problem", Technical Report LCDS M-82-7, Lefschetz Center for Dynamical Systems, Brown University, Providence RI (1982).

DAFERMOS, S.C., "The General Multimodal Network Equilibrium Problem with Elastic Demand". Lefschetz Centre for Dynamical Systems, Brown University (1980). To appear in Networks.

DABANZO, C,F, , "Unconstrained Extremal Formulations of some Transportation Equilibrium Problems" Technical Report, University of California, Berkeley (1981).

DELORME, L. and ROUSSEAU, J.-M., "The Multiregional Equilibrium Model for Resource Allocation in a Health Care System: An Application of Multiple Commodity Spatial Price Equilibrium Models". Publication #438, Dép. d'informatique et de r.o., Université de Montréal (1982).

DEMBO, R. and KLINCEWICZ, J., "A Scale Reduced Gradient Algorithm for Network Flow Problems with Convex Separable Costs". Working paper 321, Yale School of Organization and Management, New Haven, CT (1978), published in Mathematical Programming Study 15, 125-147 (1981)

DEMBO, R.S., "Progress in Large Scale Nonlinear Optimization". Presented at XI International Symposium in Mathematical Programming, Bonn W.Germany (1982).

DEMING, W.E. and STEPHAN, F.F., "On a Least Squares Adjustment of a Sampled Frequency Table when the Expected Marginal Totals are Known". Annals of Mathematical Statistics 11, 427-444 (1940).

DIAL, R.B., "A Probabilistic Multipath Assignment Model which Obviates Path Enumeration". Transportation Research 8, 3585-96 (1971).

DOW, P. and VAN VLIET, D., "Capacity Restrained Road Assignment". Traffic Engineering and Control 261-273 (1979).

ERICKSON, R.E., MANNA, C.L. and VEINOTT, A.F., "Minimum-Concave Cost Network Flows". Technical report (1981). Forthcoming in Mathematics of Operations Research.

ERLANDER, S., NGUYEN, S. and STEWART, N., "On the Calibration of the Combined Distribution-Assignment Model". Transportation Research B 13B, 259-267 (1979).

EVANS, S.P., "Derivation and Analysis of some Models for Combining Trip Distribution and Assignment". Transportation Research 10, 37-57 (1976).

EVANS, S.P. and KIRBY, H.R., "A Three-Dimensional Furness Procedure for Calibrating Gravity Models". Transportation Research 8, 105-122 (1974).

FERLAND, J.A., "Minimum Cost Multicommodity Circulation Problem with Convex Arc-Costs". Transportation Science 8, 355-360 (1974).

FISK, C., "Some Developments in Equilibrium Traffic Assignment", Transportation Research 14B, 243-255 (1980).

FLORIAN, M., "A Traffic Equilibrium Model of Travel by Car and Public Transit Modes". Transportation Science 8, 166-179 (1977).

FLORIAN, M., "An Improved Linear Approximation Algorithm for the Network Equilibrium (Packet Switching) Problem". IEEE Proceedings, Decision and Control, 812-818 (1977).

FLORIAN, M., "An Introduction to Network Models Used in Transportation Planning" forthcoming in Transportation Planning Models, edited by M. Florian, North-Holland (1982).

FLORIAN, M. and LOS, M., "A New Look at Static Spatial Price Equilibrium Models". Regional Science and Urban Economics 12, 579-597 (1982).

FLORIAN, M. and NGUYEN, S., "A Combined Trip Distribution Modal Split and Trip Assignment Model", Transportation Research 12, 241-246 (1978).

FLORIAN, M. and NGUYEN, S., "A Method for Computing Network Equiblibrium with Elastic Demand". Transportation Science 8, 321-332 (1974).

FLORIAN, M. and NGUYEN, S., "An Application and Validation of Equilibrium Trip Assignment Methods". Transportation Science 10, 374-389 (1976).

FLORIAN, M. and SPIESS, H., "On Binary Mode Choice/Assignment Models". Transportation Science 17, 32-47 (1983).

FLORIAN, M. and SPIESS, H., "The Convergence of Diagonalization Algorithms for Asymmetric Equilibrium Problems". Transportation Research B 16B, 477-483 (1982).

FLORIAN, M., CHAPLEAU, R., NGUYEN, S., ACHIM, C., JAMES-LEFEBVRE, L. and FISK, C., "Validation and Application of EMME: An Equilibrium Based Two-Mode Urban Transportation Planning Method", Transportation Research Record 728, 14-22 (1979).

FLORIAN, M., FERLAND, J. and NGUYEN, S., "On the Combined Distribution-Assignment of Traffic". Transportation Science 9, 43-53 (1975).

FLORIAN, M., FERLAND, J. and ZUBIETA, L., A New Look at Algorithm for Static Spatial Price Equilibrium Models". Bulletin of the 1982 San Diego TIMS/ORSA Meeting.

FLORIAN, M., GUELAT, J. and SPIESS, H., "An Efficient Implementation of the PARTAN Variant of the Linear Approximation Method for the Network Equilibrium Problem" forthcoming (1983).

FLORIAN, M., ROBBIN ARTHIA, M. and DE WERRA, D., "A Property of Minimum Concave Cost Flows in Capacitated Networks". Can. J. Operational Research 9, 293-304 (1971).

FRANK, C., "A Study of Alternative Approaches to Combined Trip Distribution-Assignment Modeling". Ph.D. Thesis,, Dept of Regional Science, University of Pennsylvania (1978).

FRANK, M. and WOLFE, P., "An Algorithm for Quadratic Programming". Naval Research Logistic Quarterly 3, 95-110 (1956).

FRANK, M. and WOLFE, P., "An Algorithm of Quadratic Programming". Naval Res. Log. Quart. 3, 95-110 (1956).

FRATTA, L., GERLA, M. and KLEINROCK, L., "The Flow Deviation Method: An Approach to Store and Forward Communication Network Design". Networks 3, 97-133 (1973).

FURTRAN, W.H., NABY, J.B. and STOREY, B.B., "The Impact on the Canadian Rapeseed Industry from Changes in Transport and tariff rates" American Journal of Agriculture Economics 61, 238-248 (1979).

GALLO, G. and GODINI, C., "Adjacent Extreme Flows and Application to Min Concave Cost Flows Problems". Networks 9 (1979).

GALLO,, G., SANDI, C. and GODINI, C., "An Algorithm for the Min Concave Cost Flow Problem". European Journal of Operations Research 4, 249-255 (1980).

GIBERT, A., "A Method for the Traffic Assignment Problem when Demand Is Elastic". LBB TNT-85 (1968). Unpublished.

GLASSEY, C.R., "A Quadratic Network Optimization Model for Equiblibrium Single Commodity Travel Flows" Mathematical Programming 14, 98-104 (1978).

GOLDEN, B., "A Minimum Cost Multi-Commodity Network Flow Problem Concerning Imports and Exports" Networks 5, 331-356 (1975).

GORMAN, W.M., "Estimating Trends in Leontief Matrices". A Note on Mr. Bacharach's paper, note, Nuffield College, Oxford, duplicated (1963).

GUELAT, J., "Algorithmes pour le problème d'affectation du trafic avec demandes fixes - Comparaisons". Technical Report, Centre de recherche sur les transports, Université de Montréal (1982).

HARTWICK, J.M., "The Generalized Transportation Problem as a Quadratic Program" Discussion paper 435, Institute for

Economic Research, Queen's University, Kingston (1971).

HEARN, D.W., "The Gap Function of a Convex Program". OR Letters 1, 2 (1982).

HEARN, D.W. and NGUYEN, S., "Dual and Saddle Functions Related to the Gap Function". Research Report 82-4, Dept. of Industrial and Systems Engineering, University of Florida (1982).

HEARN, D.W., , "The Gap Function of a Convex Program". Operations Research Letters 1, 67-71 (1982)

HOGAN, W.W. , "Energy Policy Models for Project Independence" Computer and Operations Research 2 (1975).

HOLLOWAY, C.A., "An Extension of the Frank and Wolfe Method for Feasible Directions". Mathematical Programming 6, 14-227 (1974).

HUTCHINSON, B., "Land Use-Transport Planning Models" forthcoming in Transportation Planning Models, edited by M. Florian, North-Holland (1982).

IRWIN, C. and YANG, C., "Iteration and Sensitivity for a Spatial Equilibrium Problem with Linear Supply and Functions". Operations Research 30, 319-335 (1982).

JAYNES, E., "Information Theory and Statistical Mechanics". Physical Review 106, 171-190; 108, 620-630 (1957).

JEFFERSON, T.R. and SCOTT, C.H., "The Analysis of Entropy Models with Equality and Inequality Constraints". Transportation Research B 13B, 123-132 (1979).

JORNSTEN, K. and NGUYEN, S.., "On the Estimation of a Trip Matrix from Network Data". Publication #153, Centre de recherche sur les transports, Université de Montréal (1979).

JOSEPHY, N., "Newton's Method for Generalized Equations". Technical Report 1965, Mathematics Research Center, University of Wisconsin, Madison, WI 91979).

JUDGE, G.G. and TAKAYAMA, T., eds, "Studies in Economic Planning Over Space and Time" North-Holland/American Elsevier, Amsterdam (1973).

KENNEDY, M., "An Economic Model of the World Oil Market". The Bell Journal of Economics and Management Science 5 (1974).

KINDERLEHRER, D. and STAMPACCHIA, G., "An Introduction to Variational Inequalities and Applications". Academic Press (1980).

KRUITHOF, J., "Calculation of Telephone Traffic". De Ingenieur (E. Elektrotechnik 3 52, 15-25 (1937).

KULLBACK, S., Information Theory and Statistics. John Wiley, New York (1959).

LAMOND, B. and STEWART, N.F., "Bregman's Balancing Method". Transportation Research B 15B, 239-248 (1981).

LE BLANC, L., "Global Solutions for a Non Convex, Non Concave Rail Network Model". Management Science 23, 131-139 (1978).

LE BLANC, L., "Mathematical Programming Algorithm for Large Scale Network Equilibrium and Network Design Problems". Ph.D. Dissertation, Dept. of Industrial Engineering & Management Sciences, Northwestern Un., Evanston, IL (1973).

LE BLANC, L.J. and FARHANGIAN, K., "Efficient Algorithms for Solving Elastic Demand Traffic Assignment Problems and Mode Split Assignment Problems". Transportation Science 15, 306-312 (1981).

LE BLANC, L.J., MORLOK, E.K. and PIERSKALLA, W.P., "An Efficient Approach to Solving the Road Network Equilibrium Traffic Assignment Problem". Transportation Research 5, 309-318 (1975).

LE BLANC, L.J., HELGASON, R.V. and BOYCE, D.E., "Improved Efficiency of the Frank-Wolfe Algorithm". Working paper 381-131, Owen Graduate School of Management, Vanderbilt University (1982).

LEVENTHAL, T., NEMHAUSER, G. and TROTTER, J., Jr., "A Column Generation Algorithm for Optimal Traffic Assignment". Transportation Science 7, 168-176 (19).

LOS, M., "Combined Residential Location and Transportation Models" Environment and Planning A 11, 12141-1265 (1979).

LUENBERGER, D.G., Introduction to Linear and Nonlinear Programming. Addison-Wesley, Reading, MA (1973).

MAC KINNON, J.G., , "An Algorithm for the Generalized Transportation Problem" Regional Science and Urban Economics 5,445-464 (1975).

MAGNANTI, L., "Equilibria and Congested Networks: Modeling and Computation". ORSA/TIMS Bulletin 10, (1980).

MAGNANTI, T.L., "Models and Algorithms for Predicting Urban Traffic Equilibrium". Forthcoming in Transportation Planning Models, edited by M. Florian, North-Holland (1982).

MURCHLAND, J.D., , "Road Network Traffic Distribution in Equilibrium". In Proceedings of the Conference "Mathematical Methods in Economic Sciences", Mathematisches Forschungsinstitut, Oberwolfen (1969).

MURTAGH, B. and SAUNDERS, M., "MINOS: A Large-Scale Nonlinear Programming System - User's Guide", Technical report, Department of Operations Research, Stanford University (1977).

NAGY, J.G., FURTRAN, W.H. and KULSHRESHTA, S.N., "The Implication of a Change in the Crow's Nest Pass Freight Rate on the Canadian Prairie Wheat Economy", The Logistics and Transportation Review 15, 547-566 (1980).

NGUYEN, S., "A Mathematical Programming Approach to Equilibrium Methods of Traffic Assignment". Publication #138, Dép. d'informatique et de r.o., Université de Montréal (1973).

NGUYEN, S., "A Unified Approach to Equilibrium Methods for Traffic Assignment". In Traffic Equilibrium Methods, Proceedings 1974, M.Florian, ed., Vol.118 Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, 148-182 (1976).

NGUYEN, S., "An Algorithm for the Traffic Assignment Problem". Transportation Science 8, 203-216 (1974).

NGUYEN, S., "Estimating an OD Matrix from Network Datas: A Network Equilibrium Approach". Publication #60, Centre de recherche sur les transports, Université de Montréal (1977).

NGUYEN, S., Modèles de distribution spatiale tenant compte des itinéraires". Publication #225, Centre de recherche sur les transports, Université de Montréal (1981).

NGUYEN, S. and DUPUIS, C., "An Efficient Method for Computing Traffic Equilibria in Network with Asymmetric Transportation Costs". Publication #205A, Centre de recherche sur les transports, Université de Montréal (1982).

NGUYEN, S., , "Procedures for Equilibrium Traffic Assignment with Elastic Demand". Publication 339, Centre de recherche sur les transports, Université de Montréal (1977).

OBUCKI, A. and KAJI I., "An Algorithm for the Hitchcock Transportation Problems with Quadratic Cost Functions", Journal of the Operations Research Society of Japan 24, 170-191 (1981).

PANG, J. and CHAN, D., "Gauss-Seidel Methods for Variational Inequality Problems over Product Sets". Working paper, Graduate School of Industrial Administration, Carnegie-Mellon,

Pittsburgh, PA (1982).

PANG, J. and CHAN, D., "Iterative Methods for Variational and Complementarity Problems". To appear in Mathematical Programming (198).

PANG, J.S., "A Hybrid Method for the Solution of some Multi-Commodity Spatial Equilibrium Problems". Management Science 27, 1142-1157 (1981).

PANG, J.S. and LEE, P.S.C., "A Parametric Linear Complementarity Technique for the Computation of Equilibrium Prices in a Single-Commodity Spatial Model". Math. Programming 20 (1981).

POLITO, J., McCARL, B.A. and MORIN, T.L., "Solution of Spatial Equilibrium Problems with Benders' Decomposition". Management Science 26, 593-605 (1980).

POTTS, R. and OLIVER, R., Flows in Transportation Networks. Academic Press (1972).

POWELL, W.B. and SHEFFI, Y., "The Convergence of Equilibrium Algorithms with Predetermined Step Sizes". Transportation Science

RECH, P. and BARTON, L.G., "A Non-Convex Transportation Algorithm". Applications of Mathematical Programming Techniques, ed. by E.M. Beale (1970).

ROBILLARD, P. and STEWART, N.F., "Iterative Numerical Methods for Trip Distribution Problems". Transportation Research 8, 575-582 (1974).

ROWSE, J., "Solving the Generalized Transportation Problem" Regional Science and Urban Economics 11, 57-68 (1981).

ROY, B., Algèbre moderne et théorie des graphes Tome 2. Dunod, Paris (1970).

RUB, R. and BORNSTEIN, C., "Um Algoritmo Branch-and-Bound para Problemas de Localizaicao com Custos Expressos por Funicoes Escadas". Technical Report. COEPPE,f Federal University of Rio de Janeiro, Brazil (1982).

SAMUELSON, P.A., "Spatial Price Equilibrium and Linear Programming". American Economic Review 42, 283-303 (1952).

SCHMITZ, A. and BAUDEN, D.L., "A Spatial Price Analysis of the World Wheat Economy. Some Long-Run Predictions", in G.G. Judge and T. Takayama, eds., Studies in Economic Planning over Space and Time, North-Holland, Amsterdam (1973).

SHAH, B.V., BUEHLER, R.J. and KEMPTHORNE, O. , "Some Algo-

rithms for Minimizing a Function of Several Variables". [SIAM Journal of Applied Mathematics 12, 74-92 (1964).

SMITH, M.J., "Existence, Uniqueness and Stability of Traffic Equilibria". Transportation Research B 1B, 295-304 (1979).

SNICKARS, F. and WEIBULL, J.W., "A Minimum Information Principle, Theory and Practice". Regional Science and Urban Economics 7, 137-168 (1977).

SOLAND, R.M., "Optimality Facility Location with Concave Costs". Operations Research 22, 373-382 (1974).

SOUMIS, F., , "Planification d'une flotte d'avions"' Publication #133, Centre de recherche sur les transports, Université de Montréal (1978).

STEWART, N.F., "Notes on the Mathematical Structure of Equilibrium Models". Technical Report LITH-MATH-12-79-8, Dept. of Mathematics, Linkoping Institute of Technology (1979).

TAKAYAMA, T. and JUDGE, G.G., Spatial and Temporal Price and Allocation Models North-Holland, Amsterdam (1971).

TAKAYAMA, T. and JUDGE, G.G., "Alternative Spatial Equilibrium Models". Journal of Regional Science 10, 1-12 (1970). Regional Science 10, 1-12 (1970).

TAKAYAMA, T. and JUDGE, G.G., "Equilibrium among Spatially Separated Markets: A Reformulation". Econometrica 32,510-524 (1964).

WARDROP, J., "Some Theoretical Aspect of Road Traffic Research". Proceedings of the Institute of Civil Engineers, Part II, Vol. 1, 325-378 (1952).

WEINTRAUB, A. and GONZALES, J., "An Algorithm for the Traffic Assignment Problem" Networks 10, 191-210 (1980)

WILSON, A.G., "Entropy in Urban and Regional Modelling". Pion Limited, London (1970).

WILSON, A.G., "A Statistical Theory of Spatial Distribution Models". Transportation Research 1, 253-269 (1967).

WOLFE, P., "Convergence Theory in Nonlinear Programming". In Integer and Nonlinear Programming ed. by J.A. Abadie, 1-36, North-Holland (1970).

YAGED, B., Jr., "Minimum Cost Routing for Static Network Models". Networks 1, 139-172 (1971).

ZADEH, N., "On Building Minimum Cost Communication Networks

over Time". Networks4 19-34 (1974).

ZADEH, N., "On Building Minimum Cost Communication Networks". Networks 3 315-331 (1973).

ZANGWILL, W.I., Nonlinear Programming: A Unified Approach Prentice-Hall, Englewood Cliffs, NJ (1969).

ZANGWILL, W.I., "Minimum Concave Cost Flows in Certain Networks". Management Science 14, 429-450 (1968).